PLANE STRAIN IN LAMINATED ORTHOTROPIC STRUCTURES

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1. Introduction. It is well known in the theory of plane strain in orthotropic materials[3] that if F satisfies the equation

$$\gamma(\partial^4F/\partial x^4) + 2(\nu - \beta)(\partial^4F/\partial x^2\partial y^2) + \alpha(\partial^4F/\partial y^4) = 0 \tag{1}$$

where α, β, γ, and ν depend upon the elastic constants of the material, then the stresses are given by

$$\tau_{xx} = \partial^2F/\partial y^2, \quad \tau_{yy} = \partial^2F/\partial x^2, \quad \tau_{xy} = -\partial^2F/\partial x\partial y. \tag{2}$$

The solution must also satisfy the boundary conditions at the surface. These will be called the surface boundary conditions. For laminated material it is also required that the normal and shear stresses, and the displacements, be continuous at every point on the surfaces separating the layers. These will be called internal boundary stress conditions and internal boundary displacement conditions respectively.

The strain components are defined by

$$e_{xx} = \partial u/\partial x, \quad e_{yy} = \partial v/\partial y, \quad e_{xy} = \tfrac{1}{2}(\partial u/\partial y + \partial v/\partial x). \tag{3}$$

where u and v are the displacements in the x and y directions respectively.

Using E_i for the Young's moduli, σ_{ij} for the Poisson's ratios, and μ_{ij} for the moduli of rigidity, Hooke's law can be written

$$
\begin{aligned}
e_{xx} &= \alpha\tau_{xx} - \beta\tau_{yy} \\
e_{yy} &= -\beta\tau_{xx} + \gamma\tau_{yy} \\
e_{xy} &= \nu\tau_{xy}
\end{aligned}
\qquad (4)
$$

where

$$
\begin{aligned}
\alpha &= (1 - \sigma_{xz}\sigma_{zx})/E_x \\
\beta &= (\sigma_{xy} + \sigma_{xz}\sigma_{zy})/E_x = (\sigma_{yx} + \sigma_{yz}\sigma_{zx})/E_y \\
\gamma &= (1 - \sigma_{yz}\sigma_{zy})/E_y \\
\nu &= 1/2\mu_{xy}.
\end{aligned}
\qquad (5)
$$

Solving (4) for τ_{xx}, τ_{yy}, and τ_{xy} gives

$$
\begin{aligned}
\tau_{xx} &= \omega e_{xx} + \phi e_{yy} \\
\tau_{yy} &= \phi e_{xx} + \lambda e_{yy} \\
\tau_{xy} &= 2\mu_{xy} e_{xy}
\end{aligned}
\qquad (6)
$$

where

$$
\begin{aligned}
\lambda &= \alpha/(\alpha\gamma - \beta^2) \\
\phi &= \beta/(\alpha\gamma - \beta^2) \\
\omega &= \gamma/(\alpha\gamma - \beta^2).
\end{aligned}
\qquad (7)
$$

Let $\eta = (\gamma/\alpha)^{1/4} y$ and $\kappa = (\nu - \beta)/(\alpha\gamma)^{1/2}$, then equation (1) can be written

$$\partial^4 F/\partial x^4 + 2\kappa(\partial^4 F/\partial x^2 \partial\eta^2) + \partial^4 F/\partial\eta^4 = 0. \qquad (8)$$

This equation is satisfied by

$$F = R(F_1(z_1) + F_2(z_2)) \qquad (9)$$

where

$$z_1 = x + i\delta_1 y = x + i[\kappa + (\kappa^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}}\eta$$

$$z_2 = x + i\delta_2 y = x + i[\kappa - (\kappa^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}}\eta \tag{10}$$

with

$$\delta_1 = (\gamma/\alpha)^{\frac{1}{4}}[\kappa + (\kappa^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$\delta_2 = (\gamma/\alpha)^{\frac{1}{4}}[\kappa - (\kappa^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}}. \tag{11}$$

If we let R_i denote the real part of $d^2F_i/dz_i{}^2$, and I the imaginary part, we have

$$\tau_{xx} = -\delta_1^2 R_1 - \delta_2^2 R_2$$

$$\tau_{yy} = R_1 + R_2$$

$$\tau_{xy} = \delta_1 I_1 + \delta_2 I_2 \tag{12}$$

and

$$e_{xx} = -\epsilon_1 R_1 - \epsilon_2 R_2$$

$$e_{yy} = \zeta_1 R_1 + \zeta_2 R_2$$

$$e_{xy} = \nu\delta_1 I_1 + \nu\delta_2 I_2 \tag{13}$$

where $\epsilon_1 = \alpha\delta_1^2 + \beta$ and $\zeta_1 = \beta\delta_1^2 + \gamma$. (14)

Since

$$\delta_1^2\epsilon_1 + \zeta_1 - 2\nu\delta_1^2 = \alpha\delta_1^4 + 2\beta\delta_1^2 + \gamma - 2\delta_1^2\nu$$

$$= \alpha\delta_1^2 - 2(\nu - \beta)\delta_1^2 + \gamma = 0,$$

we have $\zeta_1 = \delta_1^2(2\nu - \epsilon_1)$. (15)

and also

$$\delta_1^2 \delta_2^2 = \gamma/\alpha. \tag{16}$$

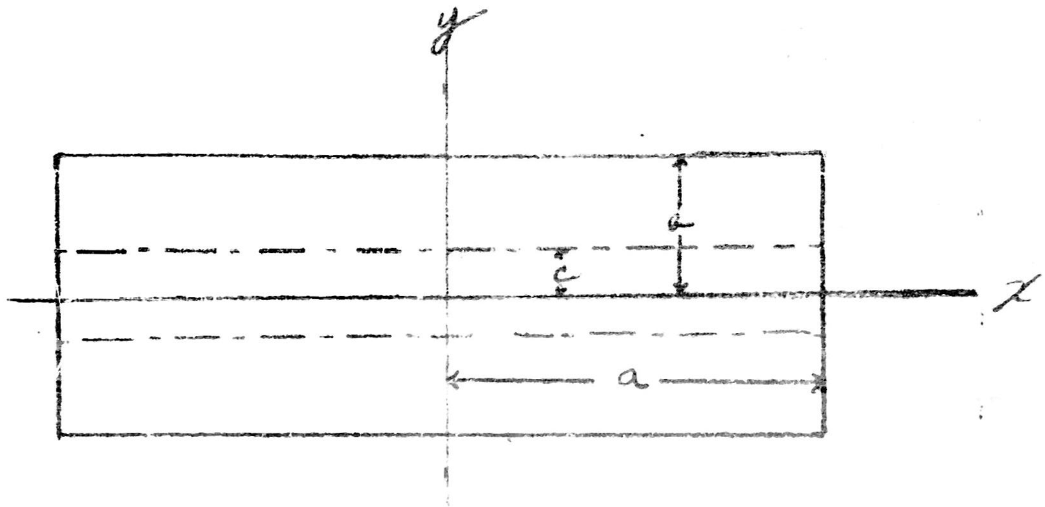
$$\delta_1^2 + \delta_2^2 = 2(\nu - \beta)/\alpha. \tag{17}$$

In this paper we consider rectangles consisting of three layers cemented together. The two outer layers, or faces, are similar and of thickness (b-c). The thickness of the inner layer, or core, is 2c. The width of each layer is 2a. The planes of separation of the layers are at $y = \pm c$. Since the problem is one of plane strain, the length of the body in the z direction is considered to be infinite. We put b=nc and a=wc.

The internal boundary stress conditions are that τ_{yy} and τ_{xy} be continuous at $y = \pm c$. The internal boundary displacement conditions are that u and v be continuous at $y = \pm c$. In part 2 we see that these internal boundary conditions eliminate many solutions of equation (8).

In part 3 we give a method of obtaining approximate solutions which satisfy surface boundary conditions on the stresses, or surface boundary conditions in which any displacements which are given are specified to be zero. This method is based on the concept of function space.

In part 4 the equation of elasticity is replaced by finite equations to give a method of obtaining approximate solutions which satisfy surface boundary conditions on the stresses.



For an example we take the material to be spruce with the longitudinal axis in the direction of the x-axis in the middle layer and the transverse axis in the direction of the x-axis in the outer layers, the radial axis being in the direction of the y-axis in both layers. We take the constants of the material to be as follows:

$$E_L = 1.95\cdot 10^6 \text{ lb/in}^2 \qquad E_R = 0.128\cdot 10^6 \text{ lb/in}^2 \qquad E_T = 0.0691\cdot 10^6 \text{ lb/in}^2$$

$$\mu_{LR} = 0.104\cdot 10^6 \text{ lb/in}^2 \qquad \mu_{LT} = 0.0720\cdot 10^6 \text{ lb/in}^2 \qquad \mu_{RT} = 0.0045\cdot 10^6 \text{ lb/in}^2$$

$$\sigma_{LR} = 0.450 \qquad \sigma_{RT} = 0.559 \qquad \sigma_{TL} = 0.0194$$

$$\sigma_{RL} = 0.0300 \qquad \sigma_{TR} = 0.301 \qquad \sigma_{LT} = 0.539 \qquad (18)$$

Using the values given in (18) we calculate:

$\alpha_1 = 0.507\cdot10^{-6}$ in²/lb $\qquad \alpha_2 = 14.3\cdot10^{-6}$ in²/lb

$\beta_1 = 0.314\cdot10^{-6}$ in²/lb $\qquad \beta_2 = 4.48\cdot10^{-6}$ in²/lb

$\gamma_1 = 6.48\cdot10^{-6}$ in²/lb $\qquad \gamma_2 = 7.69\cdot10^{-6}$ in²/lb

$\nu_1 = 4.81\cdot10^{-6}$ in²/lb $\qquad \nu_2 = 108.7\cdot10^{-6}$ in²/lb

$\delta_{11} = 4.12 \qquad \delta_{12} = 3.81$

$\delta_{21} = 0.868 \qquad \delta_{22} = 0.192$

$\epsilon_{11} = 8.92\cdot10^{-6}$ in²/lb $\qquad \epsilon_{12} = 212\cdot10^{-6}$ in²/lb

$\epsilon_{21} = 0.696\cdot10^{-6}$ in²/lb $\qquad \epsilon_{22} = 5.01\cdot10^{-6}$ in²/lb

$\zeta_{11} = 11.8\cdot10^{-6}$ in²/lb $\qquad \zeta_{12} = 72.8\cdot10^{-6}$ in²/lb

$\zeta_{21} = 6.72\cdot10^{-6}$ in²/lb $\qquad \zeta_{22} = 7.85\cdot10^{-6}$ in²/lb

2. Exact Polynomial Solutions. Let

$$F_1 = \sum_{n=0}^{m} (P_n + iM_n) z_1^{n+2}$$

$$F_2 = \sum_{n=0}^{m} (Q_n + iN_n) z_2^{n+2}. \qquad (1)$$

Then, indicating the layer by means of a second subscript, with 1 for the core and 2 for the faces, the internal boundary stress conditions can be written

$$\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-k} (-1)^{\frac{1}{2}p} (k+p+2)(k+p+1) \binom{k+p}{k} c^p (\delta_{11}{}^p P_{k+p,1} + \delta_{21}{}^p Q_{k+p,1} - \delta_{12}{}^p P_{k+p,2} - \delta_{22}{}^p Q_{k+p,2}) = 0 \qquad (2)$$

$$\sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-k} (-1)^{\frac{1}{2}p-\frac{1}{2}} (k+p+2)(k+p+1) \binom{k+p}{k} c^p (\delta_{11}{}^{p+1} P_{k+p,1} + \delta_{21}{}^{p+1} Q_{k+p,1} - \delta_{12}{}^{p+1} P_{k+p,2} - \delta_{22}{}^{p+1} Q_{k+p,2}) = 0 \qquad (3)$$

$$\sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-k} (-1)^{\frac{1}{2}p-\frac{1}{2}} (k+p+2)(k+p+1) \binom{k+p}{k} c^p (\delta_{11}{}^p M_{k+p,1} + \delta_{21}{}^p N_{k+p,1} - \delta_{12}{}^p M_{k+p,2} - \delta_{22}{}^p N_{k+p,2}) = 0 \qquad (4)$$

$$\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-k} (-1)^{\frac{1}{2}p} (k+p+2)(k+p+1) \binom{k+p}{k} c^p (\delta_{11}{}^{p+1} M_{k+p,1} + \delta_{21}{}^{p+1} N_{k+p,1} - \delta_{12}{}^{p+1} M_{k+p,2} - \delta_{22}{}^{p+1} N_{k+p,2}) = 0. \qquad (5)$$

The internal boundary displacement conditions can be written

$$\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-k} (-1)^{\frac{1}{2}p}(k+p+2)\binom{k+p+1}{k+1}c^p(\delta_{11}{}^p\epsilon_{11}P_{k+p,1} + \delta_{21}{}^p\epsilon_{21}Q_{k+p,1} - \delta_{12}{}^p\epsilon_{12}P_{k+p,2} - \delta_{22}{}^p\epsilon_{22}Q_{k+p,2}) = 0 \qquad (6)$$

$$\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-k} (-1)^{\frac{1}{2}p}(k+p+2)\binom{k+p+1}{k}c^p(\delta_{11}{}^p\sigma_{11}P_{k+p,1} + \delta_{21}{}^p\sigma_{21}Q_{k+p,1} - \delta_{12}{}^p\sigma_{12}P_{k+p,2} - \delta_{22}{}^p\sigma_{22}Q_{k+p,2}) = 0 \qquad (7)$$

$$\sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m} (-1)^{\frac{1}{2}p-\frac{1}{2}}(p+2)c^p(\delta_{11}{}^{p+1}\epsilon_{11}P_{p,1} + \delta_{21}{}^{p+1}\epsilon_{21}Q_{p,1} - \delta_{12}{}^{p+1}\epsilon_{12}P_{p,2} - \delta_{22}{}^{p+1}\epsilon_{22}Q_{p,2}) = 0 \qquad (8)$$

$$\sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-k} (-1)^{\frac{1}{2}p-\frac{1}{2}}(k+p+2)\binom{k+p+1}{k+1}c^p(\delta_{11}{}^p\epsilon_{11}M_{k+p,1} + \delta_{21}{}^p\epsilon_{21}N_{k+p,1} - \delta_{12}{}^p\epsilon_{12}M_{k+p,2} - \delta_{22}{}^p\epsilon_{22}N_{k+p,2}) = 0 \qquad (9)$$

$$\sum_{\substack{p=-1 \\ p \text{ odd}}}^{\leq m-k} (-1)^{\frac{1}{2}p-\frac{1}{2}}(k+p+2)\binom{k+p+1}{k}c^p(\delta_{11}{}^p\sigma_{11}M_{k+p,1} + \delta_{21}{}^p\sigma_{21}N_{k+p,1} - \delta_{12}{}^p\sigma_{12}M_{k+p,2} - \delta_{22}{}^p\sigma_{22}N_{k+p,2}) = 0 \qquad (10)$$

$$\sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m} (-1)^{\frac{1}{2}p-\frac{1}{2}}(p+2)c^p(\delta_{11}{}^p\sigma_{11}M_{p,1} + \delta_{21}{}^p\sigma_{21}N_{p,1} - \delta_{12}{}^p\sigma_{12}M_{p,2} - \delta_{22}{}^p\sigma_{22}N_{p,2}) = 0. \qquad (11)$$

In equations (2)-(7) the range of k is given by $0 \leq k \leq m$, in equation (9) it is given by $-1 \leq k \leq m-1$, and in equation (10) by $1 \leq k \leq m+1$.

Equations (1) - (11) have no non-trivial solutions for $m > 2$.

For $m = 2$, we get

$$P_{ij} = Q_{ij} = M_{ij} = N_{ij} = 0, \text{ for } i = 1, 2;\ j = 1, 2.$$

$$P_{01} = \frac{\delta_{22}^2 - \delta_{12}^2}{\delta_{21}^2 - \delta_{11}^2} \begin{vmatrix} \alpha_2 & \epsilon_{21} - \beta_2 \\ \beta_2 & \delta_{21} - \gamma_2 \end{vmatrix} \frac{Q_{02}}{\Delta}$$

$$Q_{01} = -\frac{\delta_{22}^2 - \delta_{12}^2}{\delta_{21}^2 - \delta_{11}^2} \begin{vmatrix} \alpha_2 & \epsilon_{11} - \beta_2 \\ \beta_2 & \delta_{11} - \gamma_2 \end{vmatrix} \frac{Q_{02}}{\Delta}$$

$$P_{02} = -\begin{vmatrix} \alpha_1 & \epsilon_{22} - \beta_1 \\ \beta_1 & \delta_{22} - \gamma_1 \end{vmatrix} \frac{Q_{02}}{\Delta} \qquad \text{where } \Delta = \begin{vmatrix} \alpha_1 & \epsilon_{12} - \beta_1 \\ \beta_1 & \delta_{12} - \gamma_1 \end{vmatrix}$$

$$M_{21} = -\frac{\delta_{22}}{\delta_{11}} \frac{\epsilon_{22} - \epsilon_{12}}{\epsilon_{21} - \epsilon_{11}} N_{22}, \quad N_{21} = \frac{\delta_{22}}{\delta_{21}} \frac{\epsilon_{22} - \epsilon_{12}}{\epsilon_{21} - \epsilon_{11}} N_{22}, \quad M_{22} = -\frac{\delta_{22}}{\delta_{12}} N_{22}$$

$$M_{02} = -\frac{\delta_{22}}{\delta_{12}} N_{02} - 2e^2 \frac{\delta_{22}}{\delta_{12}} \frac{\epsilon_{22} - \epsilon_{12}}{\nu_1 - \nu_2} \left(\frac{2\nu_1 - \beta_1}{\alpha_1} - \frac{3\nu_1 - \nu_2 - \beta_2}{\alpha_2} \right) N_{22}$$

$$N_{01} = \frac{\delta_{22}}{\delta_{21}} \frac{\epsilon_{22} - \epsilon_{12}}{\epsilon_{21} - \epsilon_{11}} (N_{02} + K_1 N_{22})$$

$$M_{01} = -\frac{\delta_{22}}{\delta_{11}} \frac{\epsilon_{22} - \epsilon_{12}}{\epsilon_{21} - \epsilon_{11}} (N_{02} + K_2 N_{22})$$

where

$$K_1 = 2e^2 \left(\frac{2\nu_1 - \beta_1 - 3\epsilon_{11}}{\alpha_1} - \frac{\epsilon_{12} - \epsilon_{11}}{\nu_1 - \nu_2} \frac{2\nu_1 - \beta_1}{\alpha_1} - \frac{2\nu_2 - \beta_2 - 3\epsilon_{11}}{\alpha_2} + \frac{\epsilon_{12} - \epsilon_{11}}{\nu_1 - \nu_2} \frac{3\nu_1 - \nu_2 - \beta_2}{\alpha_2} \right)$$

$$\tau_{xx1} = -2\,\frac{\epsilon_{22}-\epsilon_{12}}{\alpha_2}\begin{vmatrix}\alpha_2 & \beta_2-\beta_1\\ \beta_2 & \gamma_2-\gamma_1\end{vmatrix}\frac{Q_{02}}{\Delta} + 24\,\delta_{22}\,\frac{\epsilon_{22}-\epsilon_{12}}{\alpha_1}\,N_{22}xy$$

$$\tau_{xx2} = 2\,\frac{\epsilon_{22}-\epsilon_{12}}{\alpha_2}\begin{vmatrix}\alpha_1 & \epsilon_{11}-\beta_2\\ \beta_1 & \zeta_{11}-\gamma_2\end{vmatrix}\frac{Q_{02}}{\Delta} + 24\,\delta_{22}\,\frac{\epsilon_{22}-\epsilon_{12}}{\alpha_2}\,N_{22}xy$$

$$\tau_{yy1} = \tau_{yy2} = -2\,\frac{\epsilon_{22}-\epsilon_{12}}{\alpha_2}\begin{vmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{vmatrix}\frac{Q_{02}}{\Delta}$$

$$\tau_{xy1} = 4e^2\delta_{22}\,\frac{\epsilon_{22}-\epsilon_{12}}{\nu_1-\nu_2}\left(\frac{\beta_1+\nu_1\cdot\ \ {}_2}{\alpha_1} - \frac{\beta_2-2\nu_2}{\alpha_2}\right)N_{22} - 12\delta_{22}\,\frac{\epsilon_{22}-\epsilon_{12}}{\alpha_1}N_{22}y^2$$

$$\tau_{xy2} = 4e^2\delta_{22}\,\frac{\epsilon_{22}-\epsilon_{12}}{\nu_1-\nu_2}\left(\frac{\beta_1-2\nu_1}{\alpha_1} - \frac{\beta_2+\nu_2-3\nu_1}{\alpha_2}\right)N_{22} - 12\delta_{22}\,\frac{\epsilon_{22}-\epsilon_{12}}{\alpha_2}N_{22}y^2$$

For the given numerical example:

$M_{01} = -\,1.18N_{02} - 120e^2N_{22}$, $\qquad N_{01} = 5.59N_{02} + 22.4e^2N_{22}$

$M_{02} = -\,0.0505N_{02} - 5.08e^2N_{22}$, $\qquad M_{21} = -\,1.18N_{22}$

$N_{21} = 5.59N_{22}$, $\qquad M_{22} = -\,0.0505N_{22}$

$\tau_{xx1} = 1.24Q_{02} - 1890N_{22}xy$, $\qquad \tau_{xx2} = 0.612Q_{02} - 66.8N_{22}xy$

$\tau_{yy1} = \tau_{yy2} = 1.95Q_{02}$

$\tau_{xy1} = -\,948e^2N_{22} + 943N_{22}y^2$ $\qquad \tau_{xy2} = -\,38.7e^2N_{22} + 33.4N_{22}y^2$

3. Function Space Approximation. Since so few polynomials give exact solutions, we must have recourse to approximation. One method of obtaining approximate solutions is by use of function space.[4] A vector, $\vec{P}$, in the function space is the ordered set of stress functions, $(\tau_{xx}, \tau_{yy}, \tau_{xy})$. The scalar product of two vectors is given by

$$\vec{P}\cdot\vec{P}' = \int(e_{xx}\tau_{xx}' + e_{yy}\tau_{yy}' + 2e_{xy}\tau_{xy}')dv$$

$$= \int(e_{xx}'\tau_{xx} + e_{yy}'\tau_{yy} + 2e_{xy}'\tau_{xy})dv \qquad (1)$$

the integration being over the volume of the body.

The stress state of an exact solution is called a natural state and is denoted by $\vec{S}$.

If the boundary conditions are conditions on the stresses, or if such displacements as are specified are zero, the problem is said to be one with stress boundary conditions. In this case we have also the following notation:

$\vec{S}*$, the completely associated state, satisfies the equations of equilibrium, the surface boundary conditions on the stresses, and the internal boundary stress conditions.

$\vec{S}_p'$ $(p=1,\ldots,m)$, the homogeneous associated states, satisfy the equations of equilibrium, the internal boundary stress conditions, and have zero stresses at points on the boundary where the stresses are specified.

$\vec{I}_p'$, the orthonormal homogeneous associated states, are obtained by orthonormalizeing the set of $\vec{S}_p'$.[5]

$\vec{S}_q''$ $(q=1,\ldots,n)$, the complementary states, satisfy the equation of compatibility, the internal boundary displacement conditions, and give zero displacements at points on the boundary where zero displacements are specified.

$\vec{I}_q''$, the orthonormal complementary states, are obtained by orthonormalizing the set of $\vec{S}_q''$.

The homogeneous associated states form a linear manifold $\mathcal{M}$ called the homogeneous associated manifold. If any homogeneous associated manifold be added to the completely associated state, the result is a completely associated state. The subspace of completely associated states, $\mathcal{M}^*$, is called the completely associated subspace. The complementary states form a linear manifold, $\mathcal{N}$, called the complementary manifold. $\mathcal{M}^*$ and $\mathcal{N}$ are mutually orthogonal. Every sphere which has for its diameter a line joining a point of $\mathcal{M}^*$ to a point of $\mathcal{N}$ passes through $\vec{S}$.

Obviously, the uncertainity in the position of $\vec{S}$ will be least if we pick a point of $\mathcal{M}^*$ and of $\mathcal{N}$ such that the distance between them is a minimum. These points are given by

$$\vec{V}_{m^*} = \vec{S}^* - \sum_{p=1}^{m}(\vec{S}^*\cdot\vec{I}_p')\vec{I}_p'$$

$$\vec{V}_n'' = \sum_{q=1}^{n}(\vec{S}^*\cdot\vec{I}_q'')\vec{I}_q''. \tag{2}$$

The maximum possible error will be least if we choose as an approximation for $\vec{S}$, the center of the sphere, given by

$$\vec{C} = \tfrac{1}{2}(\vec{V}_{m^*} + \vec{V}_n''). \tag{3}$$

As a completely associated state we can take

$$\vec{S}^* = (\tau_{xx}^*,\ \tau_{yy}^*,\ \tau_{xy}^*) = (\partial^2 F^*/\partial y^2,\ \partial^2 F^*/\partial x^2,\ -\ \partial^2 F^*/\partial x\partial y) \quad (4)$$

where

$$F^* = R\{F_1^*(z_1) + F_2^*(z_2)\}$$

with

$$F_1^* = \sum_{n=0}^{m}(P_n^* + iM_n^*)z_1^{n+2}$$

$$F_2^* = \sum_{n=0}^{m}(Q_n^* + iN_n^*)z_2^{n+2}.$$

the P_n^*, Q_n^*, M_n^*, and N_n^* satisfying the equations (2-2) to (2-5).

Similarly for our first complementary state we can take

$$\vec{S}_1'' = (\tau_{1xx}'',\ \tau_{1yy}'',\ \tau_{1xy}'') = (\partial^2 F_1''/\partial y^2,\ \partial^2 F_1''/\partial x^2,\ -\ \partial^2 F_1''/\partial x\partial y) \quad (5)$$

where F_1'' is defined in a manner analogous to F^* except that equations (2-6) to (2-11) must be satisfied.

Then

$$\vec{V}_0^* = \vec{S}^*$$

$$\vec{V}_1'' = (\vec{S}^*\cdot\vec{I}_1'')\vec{I}_1'' = (\vec{S}^*\cdot\vec{S}_1'')\vec{S}_1''/\vec{S}_1''\cdot\vec{S}_1'' \quad (6)$$

Using equations (1-12) and (1-13) we obtain

$$\vec{S}^*\cdot\vec{S}_1'' = \int [\delta_1^2\epsilon_1 R_1^* R_1'' + \delta_1^2\epsilon_2 R_1^* R_2'' + \delta_2^2\epsilon_1 R_2^* R_1'' + \delta_2^2\epsilon_2 R_2^* R_2'' + \zeta_1 R_1^* R_1''$$

$$+ \zeta_2 R_1^* R_2'' + \zeta_1 R_2^* R_1'' + \zeta_2 R_2^* R_2'' + 2\nu(\delta_1^2 I_1^* I_1'' + \delta_1\delta_2 I_1^* I_2''$$

$$+ \delta_1\delta_2 I_2^* I_1'' + \delta_2^2 I_2^* I_2'')]dv \quad (7)$$

Using (1-15), (1-17), and (1-7), equation (7) can be written

$$\vec{S}^* \cdot \vec{S}_1'' = \int 2\nu\Big[\delta_1^2(R_1^* R_1'' + I_1^* I_1'') + \delta_2^2(R_2^* R_2'' + I_2^* I_2'') + \Big(\frac{\beta}{\alpha}+\frac{1}{\nu\lambda}\Big)(R_1^* R_2'' + R_2^* R_1'') + \delta_1\delta_2(I_1^* I_2'' + I_2^* I_1'')\Big]dv \qquad (8)$$

In a similar manner we obtain

$$\vec{S}^* \cdot \vec{S}^* = \int 2\nu\Big[\delta_1^2(R_1^{*2} + I_1^{*2}) + \delta_2^2(R_2^{*2} + I_2^{*2}) + 2\Big(\frac{\beta}{\alpha}+\frac{1}{\nu\lambda}\Big)R_1^* R_2^* + 2\delta_1\delta_2 I_1^* I_2^*\Big]dv \qquad (9)$$

and

$$\vec{S}_1'' \cdot \vec{S}_1'' = \int 2\nu\Big[\delta_1^2(R_1''^2 + I_1''^2) + \delta_2^2(R_2''^2 + I_2''^2) + 2\Big(\frac{\beta}{\alpha}+\frac{1}{\nu\lambda}\Big)R_1'' R_2'' + 2\delta_1\delta_2 I_1'' I_2''\Big]dv \qquad (10)$$

If we take $\frac{1}{2}(\vec{V}_0^* + \vec{V}_1'')$ as our approximation for $\vec{S}$, the maximum error is $\frac{1}{2}[\vec{S}^*\cdot\vec{S}^* - (\vec{S}^*\cdot\vec{S}_1'')^2/\vec{S}_1''\cdot\vec{S}_1'']^{\frac{1}{2}}$.

Carrying out the integration over the rectangle described on page 4, we get

$$\vec{S}^*\cdot\vec{S}_1'' = 8we^2 \underline{\sum_{k=0}^{m}\ \sum_{t=0}^{m}}\ \underline{\sum_{p=0}^{\le m-k}\ \sum_{q=0}^{\le m-t}} (-1)^{1\frac{1}{2}p+\frac{1}{2}q}\theta(k,t;p,q)w^{k+t}$$

$$\Big\{\nu_1\Big[\delta_{11}^{p+q}(G_{k+p,1}^* G_{t+q,1}'' + T_{k+p,1}^* T_{t+q,1}'') + \delta_{21}^{p+q}(H_{k+p,1}^* H_{t+q,1}''$$

$$+ U_{k+p,1}^* U_{t+q,1}'') + \Gamma_1(p,q)(G_{k+p,1}^* H_{t+q,1}'' + H_{t+q,1}^* G_{k+p,1}'')$$

$$+ \Lambda_1(p,q)(T_{k+p,1}^* U_{t+q,1}'' + U_{t+q,1}^* T_{k+p,1}'')\Big]$$

$$+ \nu_2(n^{p+q+1}-1)\Big[\delta_{12}^{p+q}(G_{k+p,2}^* G_{t+q,2}'' + T_{k+p,2}^* T_{t+q,2}'')$$

$$+ \delta_{22}^{p+q}(H_{k+p,2}^* H_{t+q,2}'' + U_{k+p,2}^* U_{t+q,2}'') + \Gamma_2(p,q)(G_{k+p,2}^* H_{t+q,2}''$$

$$+ H_{t+q,2}^* G_{k+p,2}'') + \Lambda_2(p,q)(T_{k+p,2}^* U_{t+q,2}'' + U_{t+q,2}^* T_{k+p,2}'')\Big]\Big\}. \qquad (11)$$

We have used $\underline{\sum_{k=0}\ \sum_{t=0}}$ to indicate that in the summation k and t are to have the same parity, that is, they are both to be odd or both even. Similarly for p and q. The other new symbols used have the following meanings:

For p and q odd
$$\begin{cases} \Gamma_1(p,q) = \delta_{11}^{p}\delta_{21}^{q} \\ \Lambda_1(p,q) = \delta_{11}^{p-1}\delta_{21}^{q-1}(\beta_1/\alpha_1 + 1/\nu_1\lambda_1) \end{cases}$$

For p and q even
$$\begin{cases} \Gamma_1(p,q) = \delta_{11}^{p-1}\delta_{21}^{q-1}(\beta_1/\alpha_1 + 1/\nu_1\lambda_1) \\ \Lambda_1(p,q) = \delta_{11}^{p}\delta_{21}^{q} \end{cases}$$

$$\theta(\kappa,t;p,q) = \theta(p,q;\kappa,t) = \theta(t,\kappa;q,p) = \theta(q,p;t,\kappa)$$

$$= (\kappa+p+2)(t+q+2)\frac{\kappa+p+1}{\kappa+t+1}\,\frac{t+q+1}{p+q+1}\binom{\kappa+p}{\kappa}\binom{t+q}{t}$$

$$G_{ij} = c^i P_{ij} \qquad H_{ij} = c^i Q_{ij} \qquad T_{ij} = \delta_{1j} c^i M_{ij} \qquad U_{ij} = \delta_{2j} c^i N_{ij}$$

with the appropriate superscript, * or ". In addition, P^* and Q^* are to satisfy equations (2-2) and (2-3), M^* and N^* equations (2-4) and (2-5), P'' and Q'' equations (2-6) to (2-8), and M'' and N'' equations (2-9) to (2-11). The expressions for $\vec{S}^*\cdot\vec{S}^*$ and $\vec{S}_1''\cdot\vec{S}_1''$ can be obtained from that of $\vec{S}^*\cdot\vec{S}_1''$ by replacing " by * and * by " respectively.

Any set of F_p' such that $\tau_{pxx}' = \partial^2 F_p'/\partial y^2$, $\tau_{pyy}' = \partial^2 F_p'/\partial x^2$, and $\tau_{pxy}' = -\partial^2 F_p'/\partial x\partial y$ equal zero on the boundary and such that τ_{pyy}' and τ_{pxy}' are continuous at $y = \pm c$ will serve to determine the $\vec{S}_p'$. Such a set is given by

$$F_p' = (x^2 - a^2)^2(y^2 - b^2)^2 f_p(x,y) \tag{1A}$$

where $f_p(x,y)$ is continuous and has continuous derivatives in the closed region occupied by the body.

A set of $\vec{S}_q''$ can be constructed from any set of displacements which take on zero values at points of the boundary where the displacements are specified to be zero, and which are continuous.

It is not necessary that the F^* used to determine the completely associated state, $\vec{S}^*$, satisfy the equation (1-8). Any F^* such that $\tau_{xx}^* = \partial^2 F^*/\partial y^2$, $\tau_{yy}^* = \partial^2 F^*/\partial x^2$, and $\tau_{xy}^* = -\partial^2 F^*/\partial x \partial y$ satisfy the surface boundary conditions and such that τ_{yy}^* and τ_{xy}^* are continuous at $y = \pm c$ can be used to obtain an approximate solution.

As an example, consider the rectangle described on page 4, subject to the following boundary conditions:

I. At $y = b$, $\tau_{yy} = -\kappa$, $\tau_{xy} = 0$; at $y = -b$, $\tau_{yy} = \tau_{xy} = 0$.

II. At $x = \pm a$: (a) $\int_{-b}^{b} \tau_{xy} dy = \pm \kappa a$, (b) $\int_{-b}^{b} \tau_{xx} dy = 0$,
(c) $\int_{-b}^{b} \tau_{xx} y dy = 0$.

Conditions I are satisfied by taking $m = 3$ and putting:

$$G_{32}^* = H_{32}^* = G_{22}^* = H_{22}^* = G_{12}^* = H_{12}^* = T_{22}^* = U_{22}^* = T_{02}^* = U_{02}^* = 0.$$

$$H_{02}^* = -(\kappa/4 + G_{02}^*), \quad T_{32}^* = -U_{32}^* = \kappa/80n^3(\delta_{12}^2 - \delta_{22}^2), \quad U_{12}^* = \kappa/8n - T_{12}^*$$

From equations (2-2) to (2-5) we get:

$$G_{31}^* = H_{31}^* = G_{21}^* = H_{21}^* = G_{11}^* = H_{11}^* = T_{21}^* = U_{21}^* = T_{01}^* = U_{01}^* = 0.$$

$$H_{01}^* = -(\kappa/4 + G_{01}^*), \quad T_{31}^* = -U_{31}^* = \kappa/80n^3(\delta_{11}^2 - \delta_{21}^2), \quad U_{11}^* = \kappa/8n - T_{11}^*.$$

Conditions II are satisfied by

$$G_{02}^* = \kappa\delta_{22}^2/4(\delta_{12}^2 - \delta_{22}^2), \qquad G_{01}^* = \kappa\delta_{21}^2/4(\delta_{11}^2 - \delta_{21}^2)$$

$$T''_{12} = \left[\frac{\kappa(\delta_{12}^2 + \delta_{22}^2)(n^5 - 1)}{40n^3(n^3 - 1)} - \frac{\kappa\delta_{22}^2}{8n} - \frac{\kappa w^2}{16(n^3 - 1)}\right] / (\delta_{12}^2 - \delta_{22}^2)$$

$$T''_{11} = \left[\frac{\kappa(\delta_{11}^2 + \delta_{21}^2)}{40n^3} - \frac{\kappa\delta_{21}^2}{8n} - \frac{\kappa w^2}{16}\right] / (\delta_{11}^2 - \delta_{21}^2)$$

We choose a first complementary state which satisfies the surface boundary stress conditions I. This gives:

$$G''_{32} = H''_{32} = G''_{22} = H''_{22} = G''_{12} = H''_{12} = T''_{22} = U''_{22} = T''_{02} = U''_{02} = 0,$$

$$G''_{02} = H''_{02} = -\kappa/8, \quad T''_{32} = -U''_{32} = \kappa/80n^3(\delta_{12}^2 - \delta_{22}^2), \quad U''_{12} = \kappa/8n - T''_{12}$$

From equations (2-6) to (2-11) we get:

$$G''_{31} = H''_{31} = G''_{21} = H''_{21} = G''_{11} = H''_{11} = T''_{21} = U''_{21} = T''_{01} = U''_{01} = 0,$$

$$G''_{01} = \frac{\kappa}{8} \frac{\delta_{21}(\epsilon_{12} + \epsilon_{22}) - (\delta_{12} + \delta_{22})\epsilon_{21}}{\delta_{11}(\epsilon_{11} + \epsilon_{21}) - (\delta_{11} + \delta_{21})\epsilon_{11}}$$

$$H''_{01} = -\frac{\kappa}{8} \frac{\delta_{11}(\epsilon_{12} + \epsilon_{22}) - (\delta_{12} + \delta_{22})\epsilon_{11}}{\delta_{11}(\epsilon_{11} + \epsilon_{21}) - (\delta_{11} + \delta_{21})\epsilon_{11}}$$

$$T''_{31} = -U''_{31} = \kappa\alpha_2/80\alpha_1 n^3(\delta_{11}^2 - \delta_{21}^2)$$

$$T''_{11} = \frac{\kappa}{\delta_{11} - \delta_{21}}\left(\frac{\nu_1\delta_{22} - \nu_2\delta_{21}}{8n\nu_1} + \frac{\beta_2 d - \alpha_2 r - \delta_{21}s}{24n^3}\right)$$

$$U''_{11} = -\frac{\kappa}{\delta_{11} - \delta_{21}}\left(\frac{\nu_1\delta_{12} - \nu_2\delta_{11}}{8n\nu_1} + \frac{\beta_2 d - \alpha_2 r - \delta_{11}s}{24n^3}\right)$$

$$T''_{12} = \frac{1}{\delta_{12}^2 - \delta_{22}^2}\,\frac{\kappa}{\delta_{11} - \delta_{21}}\left(\frac{\nu_1[\epsilon_{22}(\delta_{11} - \delta_{21}) - \delta_{22}(\epsilon_{11} - \epsilon_{21})] + \nu_2(\epsilon_{11}\delta_{21} - \epsilon_{21}\delta_{11})}{8n\nu_1}\right.$$
$$\left. + \frac{d(\delta_{11} - \delta_{21}) - r(\epsilon_{11} - \epsilon_{21}) + s(\epsilon_{11}\delta_{21} - \epsilon_{21}\delta_{11})}{24n^3}\right)$$

$$d = [\alpha_2(2\nu_1-\beta_1)-\alpha_1(2\nu_2-\beta_2)]/\alpha_1, \quad s = [\alpha_2(\nu_1+\beta_1)-\alpha_1(\nu_2+\beta_2)]/\alpha_1\nu_1,$$

$$r = [\alpha_2^2\beta_1(\nu_1-\beta_1)-\alpha_1^2\beta_2(\nu_2-\beta_2)]/\alpha_1^2\alpha_2 - [\alpha_2\gamma_1-\alpha_1\gamma_2]/2\alpha_1$$

$$\delta_{1j} = \beta_2\epsilon_{1j}-\alpha_2 S_{1j}, \quad \rho_1 = \beta_1/\alpha_1 + 1/\nu_1\lambda_1.$$

Substituting these results in equation (11) gives:

$$\begin{aligned}
\frac{\vec{S}^{*}\cdot\vec{S}_1''}{32we^2} = {} & \nu_1\Big\{\delta_{11}^2G_{01}^{*}G_{01}''+\delta_{21}^2H_{01}^{*}H_{01}''+\rho_1(G_{01}^{*}H_{01}''+H_{01}^{*}G_{01}'')+3\big[(\delta_{11}^2+w^2)T_{11}^{*}T_{11}'' \\
& +(\delta_{21}^2+w^2)U_{11}^{*}U_{11}''+(\rho_1+w^2)(T_{11}^{*}U_{11}''+U_{11}^{*}T_{11}'')\big]-2\big\{3(\delta_{11}^4-w^4)(T_{11}^{*}T_{31}'' \\
& +T_{31}^{*}T_{11}'')+3(\delta_{21}^4-w^4)(U_{11}^{*}U_{31}''+U_{31}^{*}U_{11}'')+\big[3(\rho_1\delta_{11}^2-w^4)-5w^2(\rho_1-\delta_{11}^2)\big] \\
& (T_{31}^{*}U_{11}''+U_{11}^{*}T_{31}'')+\big[3(\rho_1\delta_{21}^2-w^4)-5w^2(\rho_1-\delta_{21}^2)\big](T_{11}^{*}U_{31}''+U_{31}^{*}T_{11}'')\big\} \\
& +20\big\{\big[(5/7)(\delta_{11}^6+w^6)+w^2\delta_{11}^2(\delta_{11}^2+w^2)\big]T_{31}^{*}T_{31}''+\big[(5/7)(\delta_{21}^6+w^6) \\
& +w^2\delta_{21}^2(\delta_{21}^2+w^2)\big]U_{31}^{*}U_{31}''+\big[(5/7)(\delta_{11}^2\delta_{21}^2\rho_1+w^6)+3w^2(\delta_{11}^2\delta_{21}^2+w^2\rho_1) \\
& -w^2(\rho_1+w^2)(\delta_{11}^2+\delta_{21}^2)\big](T_{31}^{*}U_{31}''+U_{31}^{*}T_{31}'')\big\}\Big\} + \nu_2\Big\{(n-1)\big[\delta_{12}^2G_{02}^{*}G_{02}'' \\
& +\delta_{22}^2H_{02}^{*}H_{02}''+\rho_2(G_{02}^{*}H_{02}''+H_{02}^{*}G_{02}'')\big]+3\big\{\big[(n^3-1)\delta_{12}^2+(n-1)w^2\big]T_{12}^{*}T_{12}'' \\
& +\big[(n^3-1)\delta_{22}^2+(n-1)w^2\big]U_{12}^{*}U_{12}''+\big[(n^3-1)\rho_2+(n-1)w^2\big](T_{12}^{*}U_{12}''+U_{12}^{*}T_{12}'')\big\} \\
& -2\big\{3\big[(n^5-1)\delta_{12}^4-(n-1)w^4\big](T_{12}^{*}T_{32}''+T_{32}^{*}T_{12}'')+3\big[(n^5-1)\delta_{22}^4-(n-1)w^4\big] \\
& (U_{12}^{*}U_{32}''+U_{32}^{*}U_{12}'')+\big(3\big[(n^5-1)\rho_2\delta_{22}^2-(n-1)w^4\big]-5w^2(n^3-1)(\rho_2-\delta_{22}^2)\big) \\
& (T_{12}^{*}U_{32}''+U_{32}^{*}T_{12}'')+\big(3\big[(n^5-1)\rho_2\delta_{12}^2-(n-1)w^4\big]-5w^2(n^3-1)(\rho_2-\delta_{12}^2)\big) \\
& (T_{32}^{*}U_{12}''+U_{12}^{*}T_{32}'')\big\}+20\Big\{\big\{(5/7)\big[(n^7-1)\delta_{12}^6+(n-1)w^6\big]+w^2\big[(n^5-1)\delta_{12}^4 \\
& +(n^3-1)\delta_{12}^2w^2\big]\big\}T_{32}^{*}T_{32}''+\big\{(5/7)\big[(n^7-1)\delta_{22}^6+(n-1)w^6\big]+w^2\big[(n^5-1)\delta_{22}^4 \\
& +(n^3-1)\delta_{22}^2w^2\big]\big\}U_{32}^{*}U_{32}''+\big\{(5/7)\big[(n^7-1)\delta_{12}^2\delta_{22}^2\rho_2+(n-1)w^6\big] \\
& +3w^2\big[(n^5-1)\delta_{12}^2\delta_{22}^2+(n^3-1)\rho_2w^2\big]-w^2\big[(n^5-1)\rho_2+(n^3-1)w^2\big](\delta_{12}^2+\delta_{22}^2)\big\} \\
& (T_{32}^{*}U_{32}''+U_{32}^{*}T_{32}'')\Big\}\Big\}.
\end{aligned}$$

Similar results can be obtained from this equation by substituting * for ", or by substituting " for *.

For spruce with ply of equal thickness (n=3), the constants on page 5 give:

$$\tau^{*}_{xx2}/k = (2.26 - 0.0433w^{2})(y/b) - 3.64(y^3/b^3) + 0.750(x^2y/b^3)$$

$$\tau^{*}_{xx1}/k = (0.295 - 1.13w^{2})(y/b) - 4.43(y^3/b^3) + 0.750(x^2y/b^3)$$

$$\tau''_{xx2}/k = 3.64 - 10.2(y/b) - 3.64(y^3/b^3) + 0.750(x^2y/b^3)$$

$$\tau''_{xx1}/k = 108 - 683(y/b) - 125(y^3/b^3) + 21.2(x^2y/b^3)$$

$$\tau^{*}_{yy2}/k = \tau^{*}_{yy1}/k = \tau''_{yy2}/k = -0.500 - 0.750(y/b) + 0.250(y^3/b^3)$$

$$\tau''_{yy1}/k = 2.10 - 17.1(y/b) + 7.05(y^3/b^3)$$

$$\tau^{*}_{xy2}/k = \tau^{*}_{xy1}/k = \tau''_{xy2}/k = 0.750(x/b) - 0.750(xy^2/b^3)$$

$$\tau''_{xy1}/k = 17.1(x/b) - 21.2(xy^2/b^3)$$

$$\frac{\vec{S}'' \cdot \vec{S}''_1}{\vec{S}''_1 \cdot \vec{S}^{*}_1} = (-0.835 + 3.37w^2 - 0.00166w^4)/(3070 + 7.19w^2 + 0.00499w^4)$$

In particular, for w=30 we get:

$$\tau_{xx2}/k = 0.406 - 19.0(y/b) - 2.22(y^3/b^3) + 45.8(x^2/a^2)(y/b)$$

$$\tau_{xx1}/k = 12.1 - 582(y/b) - 16.1(y^3/b^3) + 274(x^2/a^2)(y/b)$$

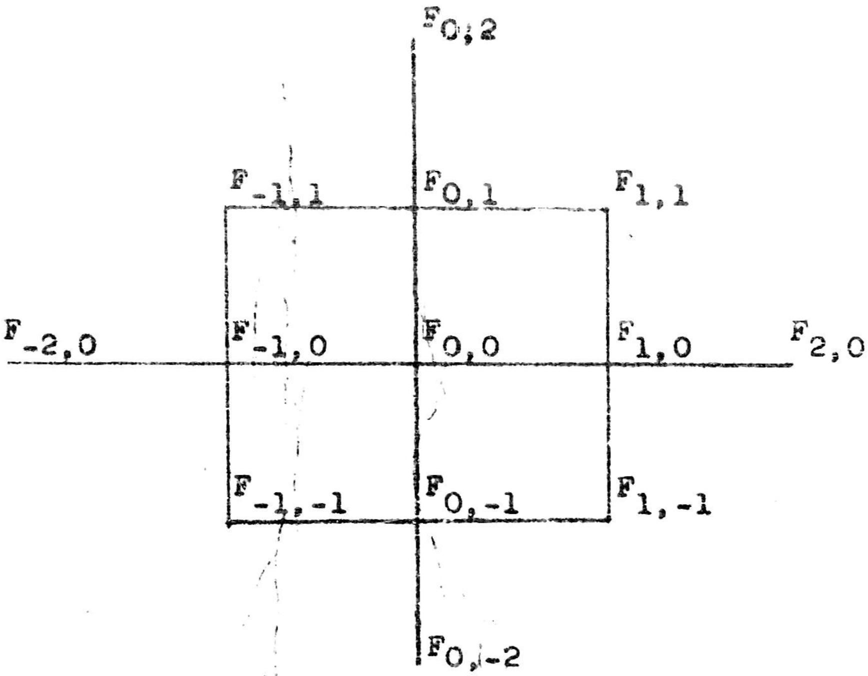
$$\tau_{yy2}/k = -0.301 - 0.469(y/b) + 0.153(y^3/b^3)$$

$$\tau_{yy1}/k = -0.0153 - 2.18(y/b) + 0.912(y^3/b^3)$$

$$\tau_{xy2}/k = 4.59(x/a) - 4.95(x/a)(y^2/b^2)$$

$$\tau_{xy1}/k = 22.8(x/a) - 27.4(x/a)(y^2/b^2)$$

4. Approximation by use of finite equations. Another method of approximation is to replace the differential equation by a set of finite equations. To do this we first approximate $F(x,y)$ by a polynomial which takes on the values $F_{ij} = F(x_i, y_j)$ at the points (x_i, y_j) as shown in (1) below. This can be done by use of Lagrange's formula of interpolation.



(1)

The stresses are then obtained from (1-2) and from them the strains are obtained by Hooke's law (1-4). These are substituted in the compatibility equation

$$\partial^2 e_{xx}/\partial y^2 + \partial^2 e_{yy}/\partial x^2 = 2(\partial^2 e_{xy}/\partial x \partial y). \qquad (2)$$

This gives

$$
\begin{aligned}
&(-144\alpha_{-2} + 96\alpha_{-1} + 900\alpha_0 + 96\alpha_1 - 144\alpha_2 - 1800\beta_0 + 804\gamma_0 \\
&+ 432\nu_{-1} + 432\nu_1)F_{0,0} + (960\beta_0 - 576\gamma_0 - 192\nu_{-1} - 192\nu_1)F_{-1,0} \\
&+ (104\alpha_{-2} - 320\alpha_{-1} - 480\alpha_0 + 64\alpha_1 + 56\alpha_2 + 384\beta_{-1} - 384\beta_0 - 320\nu_{-1} \\
&- 192\nu_1)F_{0,-1} + (960\beta_0 - 576\gamma_0 - 192\nu_{-1} - 192\nu_1)F_{1,0} + (56\alpha_{-2} \\
&+ 64\alpha_{-1} - 480\alpha_0 - 320\alpha_1 + 104\alpha_2 + 384\beta_0 + 384\beta_1 - 192\nu_{-1} \\
&- 320\nu_1)F_{0,1} + (-384\beta_0 + 160\nu_{-1} + 96\nu_1)F_{-1,-1} + (-384\beta_0 \\
&+ 160\nu_{-1} + 96\nu_1)F_{1,-1} + (-384\beta_0 + 96\nu_{-1} + 160\nu_1)F_{1,1} + (-384\beta_0 \\
&+ 96\nu_{-1} + 160\nu_1)F_{-1,1} + (-60\beta_0 + 144\gamma_0 - 24\nu_{-1} - 24\nu_1)F_{-2,0} \\
&+ (-35\alpha_{-2} + 176\alpha_{-1} + 30\alpha_0 - 16\alpha_1 - 11\alpha_2)F_{0,-2} + (-60\beta_0 + 144\gamma_0 \\
&- 24\nu_{-1} - 24\nu_1)F_{2,0} + (-11\alpha_{-2} - 16\alpha_{-1} + 30\alpha_0 + 176\alpha_1 - 35\alpha_2)F_{0,2} = 0.
\end{aligned}
$$

where α_i indicates the value of α for y --if y falls on the boundary of two regions the average of the values of α in the two regions--and similarly for β_i, γ_i, and ν_i. For a particular problem the equations (3) can conveniently be represented by a diagram similar to (1) but with the coefficients of $F_{i,j}$ written in place of $F_{i,j}$.

We calculate the value of F and its first derivatives at the boundary by integrating the values of the stresses given on the boundary. The region is divided into squares of length d by the lines $x=x_i$ and $y=y_j$. The intersections of these lines are called nodes. Nodes falling on the boundary, and, in cases where the lines intersect the boundary in points other than nodes, the nodes just inside the boundary, are called

<u>boundary nodes</u>. The nodes adjacent to the boundary nodes and exterior to the boundary are called adjacent nodes. The values of F at adjacent nodes and at boundary nodes which do not fall exactly on the boundary are approximated from the values of F and its first and second derivatives at the boundary. Nodes interior to the boundary which are not boundary nodes are called <u>interior nodes</u>. An equation can then be obtained for each interior node by considering it as the point (0, 0) in equation (3). Thus the differential equation has been replaced by the set of finite equations

$$\sum_{ij} a_{ij}^{kl} F(x_i, y_j) = 0. \tag{4}$$

The solution of these equations can be approximated by the so-called relaxation method.[6] Let F^0 be the estimated value of F at each node (taking $F^0 = F$ for a boundary node or an adjacent node). Substituting these estimated for F in the left side of (4) gives

$$\sum_{ij} a_{ij}^{kl} F^0(x_i, y_j) = R^{kl}. \tag{5}$$

If $R^{kl} = 0$ for all k and l, then $F^0 = F$ and the estimated solution is the correct solution. If this is not the case, let

$$F'(x_k, y_l) = F^0(x_k, y_l) - R'^{kl}/a_{kl}^{kl}. \tag{6}$$

The kl'th equation then becomes

$$a_{kl}^{kl} F'(x_k, y_l) + \sum_{\substack{i \neq k \\ j \neq l}} a_{ij}^{kl} F^0(x_i, y_j) = R^{kl} - R'^{kl} = R''^{kl}. \tag{7}$$

The rest of the equations which involve $F(x_k, y_l)$ become

$$a^{rs}_{kl} F'(x_k, y_l) + \sum_{\substack{i \neq k \\ j \neq l}} a^{rs}_{ij} F^{0}(x_i, y_j) = R^{rs} - a^{rs}_{kl} R'^{kl} / a^{kl}_{kl}. \qquad (8)$$

Thus by changing the estimated value of $F(x_k, y_l)$ from $F^{0}(x_k, y_l)$ to $F'(x_k, y_l)$, the "residual", R^{kl}, has been reduced to R''^{kl}. Starting with the largest $|R|$, each is relaxed in this manner until the error is negligible. The resulting F's give the approximate solution.

NOTES

[1] This research was prepared under an Office of Naval Research contract, No. [redacted])

[2] Dr. Smith is Professor of Mathematics and Dr. Blake is Assistant Professor of Mathematics at the University of Florida.

[3] H. W. March, "Bending of a centrally loaded rectangular strip of plywood", _Physics_ 7, 32-41 (1936).

[4] W. Prager and J. L. Synge, "Approximation in elasticity based on the concept of function space." _Quarterly of Applied Mathematics_ 5, 241-69 (1947).

[5] A convenient method of orthonormalizing is given in M. O. Peach, "Simplified technique for constructing orthonormal functions." _Bulletin American Mathematical Society_ 50, 556-64 (1944).

[6] R. V. Southwell, _Relaxation methods in theoretical physics_, Oxford, 1946.